THEORY OF THE TENSILE TEST*

E. W. HART†

The phenomenology of the tensile test is re-examined with special attention to the influence of strain rate sensitivity of the flow stress. Explicit formulae are deduced for the measured variables of the test in terms of the phenomenological parameters of the material. The stability of the deformation is examined, and the growth of an inhomogeneity is described in the unstable deformation regime. The effect of elasticity of the specimen and the testing machine is investigated with special attention to relaxation tests.

THEORIE DES ZUGVERSUCHS

Die Phänomenologie des Zugversuchs wird erneut untersucht, unter besonderer Berücksichtigung des Einflusses einer Verformungsgeschwindigkeitsabhängigkeit der Fließspannung. Die gemessenen Versuchsvariablen werden explizit durch die phänomenologischen Parameter des Materials ausgedrückt. Die Stabilität der Verformung wird untersucht. Die Entwicklung einer Inhomogenität wird bei der instabilen Verformung behandelt. Ferner wird der Einfluß der Elastizität von Probe und Zugapparatur untersucht, mit besonderer Berücksichtigung von Relaxationsversuchen.

THEORIE DE L'ESSAI DE TRACTION

Les phénomènes qui caractérisent l'essai de traction sont étudiés en accordant une attention particulière à l'effet de la vitesse de déformation sur le niveau de la tension de glissement. L'auteur a établi des formules caractéristiques de l'influence des différentes variables mesurables de l'essai en fonction des paramètres caractéristiques des matériaux étudiés. Il étudie également la stabilité de la déformation et la croissance d'un inhomogénéité pour un régime de déformation instable.

Il étudie enfin l'influence des caractéristiques d'élasticité, tant de l'échantillon que de la machine d'essai, sur les résultats des essais et plus particulièrement dans le cas d'essais de relaxation.

1. INTRODUCTION

This is a report of the results of a study of the phenomenology of tensile deformation of metals with special attention to the effects of deformation rate sensitivity. The investigation has been restricted to phenomena and relationships that are relatively independent of special assumptions about the microscopic mechanism of deformation. Within this limitation a number of effects can be rationalized as artifacts of the tensile or creep test, dependent only on a few measurable parameters of the material and on the mechanical properties of the testing apparatus. Specially notable among these effects is the high ductility, noted for some metals at relatively high homologous temperatures, that has been termed "superplasticity."

The importance of strain rate sensitivity at elevated temperatures was first clearly indicated by Nádai and Manjoine.[1] More recently Backofen and Avery[2] have noted its relationship to the superplasticity problem. The present work extends the conclusions of those authors in a unified manner.

The plan of this work is as follows: in section 2 the phenomenological model is introduced and the general relationships for homogeneous deformation are deduced. In Section 3 the problem of the stability of deformation in the tensile test is discussed and general criteria for stability are evaluated. In addition an estimate is made for the rate of progress of an inhomogeneity when the deformation is unstable. Section 4 contains considerations of the effect of elasticity on tensile test results when the rate sensitivity is important. The discussion in Section 5 is devoted to a critique of the tensile test.

2. RELATIONSHIPS FOR HOMOGENEOUS DEFORMATION

In order to carry out any calculations some phenomenological assumption must be made concerning the relationships among true stress σ, natural plastic strain ϵ, and strain rate $\dot{\epsilon}$. It will be assumed here that at any stage of deformation σ is dependent on the previous strain history, and that small changes in σ depend linearly on the corresponding small changes of ϵ and $\dot{\epsilon}$. Thus, in differential form,

$$d\sigma = \sigma_{\epsilon} d\epsilon + \sigma_{\dot{\epsilon}} d\dot{\epsilon}. \tag{1}$$

This differential form need not be integrable, and so the coefficients σ_{ϵ} and $\sigma_{\dot{\epsilon}}$ are not simple partial derivatives but rather should be considered as general material parameters that depend on the specimen history. These parameters change their values as the deformation proceeds, and we shall make special assumptions about their nature as needed in the subsequent treatment. A similar approach was employed earlier by Hart[3] in a consideration of Lüders bands.

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† General Electric Research and Development Center Schenectady, New York.

We shall proceed next to deduce the relationships between the measurable quantities of a tensile test and the phenomenological material parameters σ_ϵ and $\sigma_{\dot\epsilon}$ for the condition of uniform deformation.

Let P be the load supported by the specimen, L be the length of the specimen, and A be the cross-section at any instant during a deformation procedure. By these symbols we denote the instantaneous values of the quantities at each instant of the deformation history. Then, by definition of the stress,

$$P = \sigma A. \tag{2}$$

Denoting differentiation with respect to time by a dot over the symbol concerned, the time rate of change of P is readily obtained as

$$\dot P = \dot\sigma A + \sigma \dot A. \tag{3}$$

We shall represent $\dot\sigma$ by dividing equation (1) by dt, obtaining

$$\dot\sigma = \sigma_\epsilon \dot\epsilon + \sigma_{\dot\epsilon}\ddot\epsilon. \tag{4}$$

We need also the relationships

$$\dot\epsilon = \dot L/L = -\dot A/A, \tag{5}$$

and

$$\ddot\epsilon = \ddot L/L - (\dot L/L)^2 = -\ddot A/A + (\dot A/A)^2. \tag{6}$$

Now dividing equation (3) by equation (2) we obtain

$$\dot P/P = \dot A/A + \dot\sigma/\sigma,$$

and further substitutions involving equations (4–6) lead to the basic relationship

$$\dot P/P = -(\dot L/L)[1 - \gamma + m] + (\ddot L/\dot L)m, \tag{7}$$

where

$$\gamma \equiv (1/\sigma)\sigma_\epsilon, \tag{8}$$

and

$$m \equiv (\dot\epsilon/\sigma)\sigma_{\dot\epsilon}. \tag{9}$$

The quantities γ and m are material parameters and in general vary through the deformation history.

Equation (7) can now be specialized to describe two familiar types of tensile test:

Case A: The constant extension rate test.

For this test $\ddot L = 0$, and so

$$(d \ln P/d \ln L)_{\dot L} = \gamma - 1 - m, \tag{10}$$

where the subscript $\dot L$ on the derivative means that $\dot L$ is constant.

Case B: The constant load creep test.

For this case $\dot P = 0$, and since

$$(\ddot L/\dot L)/(\dot L/L) = d \ln \dot L/d \ln L,$$
$$(d \ln \dot L/d \ln L)_P = (1 - \gamma + m)/m. \tag{11}$$

Another special case of interest is the constant stress creep test. For this purpose we divide equation (4) by σ and use equations (5), (6), (8) and (9) to obtain

$$\dot\sigma/\sigma = \gamma(\dot L/L) + m[(\ddot L/\dot L) - (\dot L/L)].$$

This leads us easily to

Case C: The constant stress creep test.

For this test $\dot\sigma = 0$, and so

$$(d \ln \dot L/d \ln L)_\sigma = 1 - (\gamma/m). \tag{12}$$

3. STABILITY OF TENSILE DEFORMATION

A stability criterion

The formulas of Section 2 are valid only if the deformation proceeds homogeneously. If on the other hand a small portion of the length of the specimen has a cross-section that differs from the cross-section of the remainder by some small amount, the formulas are in error by a small amount. If in the course of the deformation the magnitude of the cross-section difference does not increase, we shall call the deformation stable. It is only when the deformation is stable or nearly so that the error can be expected to remain small. Furthermore, we can then expect high ductility without excessive localized deformation.

To investigate the stability in greater detail we shall compute the variation in area increment rate $\delta\dot A$ as a function of variation in cross-section δA. Since P is the same at all points of the specimen, a variation of equation (2) results in the relation

$$0 = \sigma\delta A + A\delta\sigma. \tag{13}$$

In analogy to equations (1), (5) and (6) we have the further equations

$$\delta\sigma = \sigma_\epsilon\delta\epsilon + \sigma_{\dot\epsilon}\delta\dot\epsilon, \tag{14}$$

$$\delta\epsilon = -\delta A/A, \tag{15}$$

$$\delta\dot\epsilon = -\delta\dot A/A + (\dot A/A)(\delta A/A). \tag{16}$$

Combining these relationships and making use of the definitions of γ and m, we obtain the result

$$(\delta \ln \dot A/\delta \ln A)_P = -(1 - \gamma - m)/m. \tag{17}$$

Now, our stability criterion can be stated as follows:

The deformation is stable if

$$(\delta\dot A/\delta A)_P \leq 0, \tag{18}$$

and is otherwise unstable. Since in tension $\dot A/A$ is negative, this condition can be restated so that the *deformation is stable in tension if*

$$(\delta \ln \dot A/\delta \ln A)_P \geq 0 \tag{18'}$$

or, in terms of the phenomenological parameters, when

$$\gamma + m \geq 1. \tag{19}$$

It is readily seen that our criterion is in agreement with classical Considère condition for strain hardening materials with negligible rate dependence, and with the usual criterion for purely viscous materials ($\gamma = 0$) for which the limiting stable material is that which displays Newtonian viscosity, or for which $m = 1$.

The constant load test

An interesting sidelight on the stability condition equation (18′) is that the point of onset of instability can be noted directly from the measured variables in the constant load creep test. This is because the comparison of a variation along the length of a specimen, since the load is the same at each point, is analogous to successive increments of deformation in time when the deformation proceeds at constant load. We may replace δA then by $\dot{A}\delta t$. The deformation at constant P is stable then so long as

$$\frac{d}{dt}\ln \dot{A} \Big/ \frac{d}{dt}\ln A \geq 0,$$

or, more simply, if

$$\ddot{A} \geq 0. \tag{20}$$

By equations (5) and (6) this leads easily to the condition

$$\ddot{L}/\dot{L} - 2(\dot{L}/L) \leq 0,$$

or

$$\ddot{L}/\dot{L} \leq 2(\dot{L}/L).$$

But the left-hand member is $d\dot{L}/dL$, and so the deformation is stable when

$$d\dot{L}/dL \leq 2(\dot{L}/L), \tag{21}$$

for creep under constant load. This condition can be applied to the plot of $\dot{L}$ versus L at constant P by simple geometrical construction.

The growth of inhomogeneities

The discussion of inhomogeneous deformation can be carried somewhat farther with the inclusion of a simplifying assumption concerning the values of the material parameters γ and m. We shall consider the progress of an inhomogeneity in a tensile specimen that is creeping under constant load during "stage two" of the deformation history. This stage is characterized grossly by a relatively constant (actually slightly increasing) $\dot{L}$, and can be more precisely described as a regime during which $\gamma = 0$ and m is constant. We need not assume anything about the nature of γ or m up to that stage.

The problem is then as follows:

> At some initial time the specimen has uniform cross-section A_0 except for a small segment that has cross-section $A_0 + \delta A_0$ where δA_0 is small.
>
> After a time interval Δt the principal cross-section has changed from A_0 to A and the special segment has become $A + \delta A$.
>
> The problem is to compute the dependence of δA upon A, A_0 and δA_0.

Note first that, corresponding to equation (17), if the variation δ is replaced by the differential d the equation holds in similar form,

$$(d \ln \dot{A}/d \ln A)_P = -(1 - \gamma - m)/m. \tag{22}$$

This can be verified also by transforming equation (11) to the form it takes when the variable L is transformed to A as variable through equations (5) and (6).

When $\gamma = 0$, equation (22) is simply

$$(d \ln \dot{A}/d \ln A)_P = 1 - \frac{1}{m}. \tag{23}$$

This equation can be integrated from A_0 to A to determine $\dot{A}$ in terms of A, A_0 and $\dot{A}_0$. The result is easily seen to be

$$\dot{A}/\dot{A}_0 = (A_0/A)^{(1/m)-1}. \tag{24}$$

If the beginning point for the integration is displaced slightly in time by an amount δt, the changes in initial and final cross-section for the same full interval Δt are respectively $\dot{A}_0\,\delta t$ and $\dot{A}\,\delta t$. Our variation δA_0 due to initial inhomogeneity can be represented by a fictitious time displacement as well and so we have that

$$\delta A/\delta A_0 = \dot{A}/\dot{A}_0,$$

and so

$$\delta A = (A_0/A)^{(1/m)-1}\delta A_0. \tag{25}$$

Therefore, if at some stage in the deformation when $\gamma = 0$, there is an inhomogeneity δA_0, after subsequent deformation, during which A_0 reduces to A, the resultant inhomogeneity δA will be given by equation (25).

The notable feature of equation (25) is the very strong dependence on m. This explains why considerable ductility can be obtained even beyond the point of instability for materials of sufficiently high m, that is for m greater than about $\frac{1}{3}$. In fact this is the principal source of the phenomenon that has been termed "superplasticity."

4. EFFECT OF ELASTICITY IN LOADING

General relationships

Since most tensile testing is done with machines that possess considerable elasticity because of connecting linkages, load measuring devices, etc. and since the specimen is also elastic, it is desirable to examine some of the phenomena that result from the elasticity. Of course this problem is not a new one, and it has been discussed by other authors before. The present treatment is in large measure an augmentation of the previous treatments with special emphasis on the rate sensitivity of the specimen.

The model that we shall take for the loading apparatus is that the specimen of length L is at any instant in series with a spring of elastic constant K that is stretched by an amount $L_1 - L$. The length L_1 is then the specimen length for which there would be no load. Then if the load is P it is given by

$$P = K(L_1 - L). \qquad (26)$$

The applied extension rate is $\dot{L}_1$, which is the extension rate that a perfectly weak specimen would experience.

We shall now consider the time derivative of equation (26)

$$\dot{P} = K(\dot{L}_1 - \dot{L}). \qquad (27)$$

If L_0 is the nominal gauge length, and A_0 the corresponding cross-section such that at any stage in the deformation,

$$LA = L_0 A_0, \qquad (28)$$

we may convert equation (27) to one involving stress and strain rates. Since

$$\dot{P} = \dot{\sigma} A + \sigma \dot{A},$$

we have

$$\dot{\sigma} A + \sigma \dot{A} = K(\dot{L}_1 - \dot{L}).$$

Then

$$\dot{\sigma} = (KL/A)[(\dot{L}_1/L) - (\dot{L}/L)] - \sigma(\dot{A}/A),$$

and since

$$\frac{\dot{A}}{A} = -\frac{\dot{L}}{L},$$

$$\dot{\sigma} = \frac{KL}{A}\left[\frac{\dot{L}_1}{L} - \left(1 - \frac{\sigma A}{KL}\right)\frac{\dot{L}}{L}\right],$$

$$= \frac{KL_0}{A_0}\left(\frac{L}{L_0}\right)^2\left[\frac{\dot{L}_1}{L} - \left(1 - \frac{\sigma A}{KL}\right)\frac{\dot{L}}{L}\right]. \qquad (29)$$

Now

$$L = L_1 - P/K,$$
$$= L_1(1 - P/KL_1).$$

But P is always much smaller than either KL or KL_1, and so we may replace L by L_1 where we wish and neglect $\sigma A/KL$ compared to 1. Then

$$\dot{\sigma} = (KL_0/A_0)(L_1/L_0)^2[(\dot{L}_1/L_1) - (\dot{L}/L)]. \qquad (30)$$

If we make the transcriptions:

$$\dot{e} \equiv \dot{L}_1/L_0,$$
$$\dot{\epsilon} \equiv \dot{L}/L,$$
$$\kappa \equiv KL_0/A_0,$$

we have finally

$$\dot{\sigma} = \kappa(L_1/L_0)^2[(L_0/L_1)\dot{e} - \dot{\epsilon}], \qquad (31)$$

where the only approximation has been to neglect σ compared to κ.

When equation (31) is applied to an interval during which the incremental strain is small, the variation of L_1 is small compared to L_1, and if L_0 is taken to be the value of L_1 at that start of the interval, the equation can be used in the simpler form

$$\dot{\sigma} = \kappa(\dot{e} - \dot{\epsilon}), \qquad (32)$$

where κ and $\dot{e}$ are computed with the same value of L_0. This is the form that is used as the starting point by Noble and Hull.[4]

Load relaxation test

If in the course of a tensile test the cross-head motion is stopped, the load will relax gradually with time. This *stress relaxation experiment* can be easily analyzed to deduce m as a function of σ as follows:

Since the relaxation is generally a slow process, each stage of the process is close to steady state. Then

$$m = d \ln \sigma / d \ln \dot{\epsilon}. \qquad (33)$$

Since $\dot{e} = 0$,

$$\dot{\sigma} = -\kappa\dot{\epsilon},$$
$$d \ln \dot{\sigma} = d \ln \dot{\epsilon},$$

and so with this geometrical constraint

$$m = (d \ln \sigma / d \ln \dot{\sigma})_{\dot{e}=0}. \qquad (34)$$

Therefore, if $\ln \sigma$ is plotted against $\ln \dot{\sigma}$, it is easy to determine m as slope of the curve at each point. Note that it is not necessary to assume any special power law for the dependence of σ on $\dot{\epsilon}$ to get this result.

General approach to steady state

If $\dot{e} \neq 0$, it is possible to estimate the approach of σ to steady state. At relatively high temperatures this is important since it is necessary to be able to distinguish a persistent change of σ due to continuing hardening of the specimen from a change of load due to elastic behavior. We shall compute the mode of approach to steady state when m is relatively constant.

From equation (32),

$$\kappa\dot{\epsilon} = \kappa\dot{e} - \dot{\sigma},$$

and

$$\ln \dot{\epsilon} = \ln \dot{e} + \ln [1 - (\dot{\sigma}/\kappa\dot{e})]. \quad (35)$$

Since near steady state $\dot{\sigma}/\kappa\dot{e} \ll 1$,

$$\ln \dot{\epsilon} \cong \ln \dot{e} - \dot{\sigma}/\kappa\dot{e},$$

and then (dropping the approximation symbol)

$$d \ln \dot{\epsilon} = -(1/\kappa\dot{e})\, d\dot{\sigma}.$$

Substituting this expression of $\dot{\epsilon}$ in equation (33), we obtain

$$-(m/\kappa\dot{e})\, d\dot{\sigma} = d \ln \sigma. \quad (36)$$

Since m, κ and $\dot{e}$ are constant in the range considered, equation (36) can be integrated.

$$-(m/\kappa\dot{e})\dot{\sigma} = \ln (\sigma/\sigma_s), \quad (37)$$

where σ_s is the steady state value of σ when $\dot{e} = \dot{\epsilon}$ and so when $\dot{\sigma} = 0$. Since $\sigma_s - \sigma \ll \sigma_s$ we can expand the right hand member of equation (37) as

$$\ln (\sigma/\sigma_s) = \ln \left[1 - \left(1 - \frac{\sigma}{\sigma_s}\right)\right],$$

$$\cong -[1 - (\sigma/\sigma_s)],$$

and write equation (37) as

$$(m\sigma_s/\kappa\dot{e})(\dot{\sigma}/\sigma_s) = 1 - \sigma/\sigma_s.$$

This is easily integrated to give

$$\sigma/\sigma_s = 1 - \exp [-(\kappa\dot{e}/m\sigma_s)(t + t_0)], \quad (38)$$

where t_0 is a constant of integration. If we make the replacements $\dot{e}t \equiv \Delta e$, $\dot{e}t_0 \equiv e_0$,

$$\sigma/\sigma_s = 1 - \exp [-(\kappa/m\sigma_s)(e_0 + \Delta e)], \quad (39)$$

where Δe is the strain variable representing the increment of strain from the starting point which must be already near the steady state.

This result shows that the approach to steady state when $\dot{e} \neq 0$ proceeds exponentially rather than by a power law which is the case for the relaxation when $\dot{e} = 0$. The characteristic strain increment that describes the speed with which σ/σ_s approaches 1 is $(m\sigma_s/\kappa)$. The smaller this number is, the more the test reflects the material properties since then $\dot{\epsilon}$ is closer to the imposed rate $\dot{e}$. That number is small when m is small and when κ/σ_s is large, i.e. for a hard machine.

5. DISCUSSION

The results of Section 3 above show that the utility of the tensile test for the determination of material properties is severely restricted for materials with small m. In particular it is not possible then to obtain reliable data in the fully hardened state where γ has become very small. It would seem that there is good reason to place greater emphasis on the use of torsion testing methods where possible. Although such methods have not been exploited much to date, there is enough experience[5] with them to indicate that they can be effectively employed, and that they can be improved by further investigations.

The main problem with tensile testing is that, because of plastic flow instability and tensile fracture behavior, it is not generally possible to obtain reliable results on the steady state stress at a given strain rate. These data are readily obtained in torsion tests even with solid specimens. The early strain hardening behavior, however, is not obtained easily in torsion tests except with tubular specimens. But that range of data is explored readily by the tensile test. It would seem therefore that a judicious combination of tensile and torsion testing could produce reliable results for most materials over a wide range of testing temperatures and rates.

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